

IV. Inclusion/exclusion

Inclusion/exclusion: Two events

- $|A \cup B| = |A| + |B| - |A \cap B|$
 - $|A \cap B| = |A| + |B| - |A \cup B|$
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Inclusion/exclusion: Three events

- $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
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Example I

At a small college, 150 freshmen are taking English, and 120 freshmen are taking math. If 90 freshmen are taking both classes, then how many are taking at least one of the two?

$$\begin{aligned} A &:= \text{English} \\ B &:= \text{math} \\ |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 150 + 120 - 90 = \boxed{180} \end{aligned}$$

Example II

At the same college, 100 freshmen are taking history. If 80 are taking both English and history, 75 are taking both math and history, and 60 are taking all three classes, then how many are taking at least one of the three? How many are taking only history?

A := English

B := math

C := history

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 150 + 120 + 100 - 90 - 80 - 75 + 60 = \boxed{185} \end{aligned}$$

Draw a Venn diagram and fill it in from the inside out to get $\boxed{40}$ people in English only, $\boxed{15}$ in math only, and $\boxed{5}$ in history only.

Example III

Student ID numbers at a small college range from 000 to 999. How many of these numbers have at least one 3 and at least one 4 in them?

$$\begin{aligned} A &:= \text{has a 3} \\ B &:= \text{has a 4} \\ A \cup B &= \text{has a 3 or a 4} \\ |A| &= 1000 - 9^3 = 1000 - 729 = 271 \\ |B| &= 1000 - 9^3 = 1000 - 729 = 271 \\ |A \cup B| &= 1000 - 8^3 = 1000 - 512 = 488 \\ |A \cap B| &= |A| + |B| - |A \cup B| \\ &= 271 + 271 - 488 = 542 - 488 = \boxed{54} \end{aligned}$$

[Alternate solution: We have three permutations of 334, and three permutations of 443, and then 6 permutations each of 34x for some other digit x, giving $3 + 3 + 48 = \boxed{54}$.]

Example IV

How many whole numbers between 1 and 1000 are divisible by 2, 3, or 5?

$$\begin{aligned} A &:= \text{divisible by 2} \\ B &:= \text{divisible by 3} \\ C &:= \text{divisible by 5} \\ |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 500 + 333 + 200 - \frac{996}{6} - 100 - \frac{990}{15} + \frac{990}{30} \\ &= 1033 - 166 - 100 - 66 + 33 = \boxed{734} \end{aligned}$$

Example V

A restaurant served 200 customers on a busy night. If 125 of them ordered appetizers, 110 ordered desserts, and 170 ordered at least one of an appetizer and dessert, then how many ordered both?

$$\begin{aligned} A &:= \text{appetizer} \\ B &:= \text{dessert} \\ |A \cap B| &= |A| + |B| - |A \cup B| \\ &= 125 + 110 - 170 = \boxed{65} \end{aligned}$$