XX. Systems of equations: complex eigenvalues

Lesson Overview

- Recall that to solve the system of linear differential equations $\mathbf{x}' = A\mathbf{x}$, we find the eigenvalues and eigenvectors of A.
- If the eigenvalues are complex, then they will occur in conjugate pairs:

$$r_1 = a + bi, r_2 = a - bi$$

- Choose one of the eigenvalues and its corresponding eigenvector $\mathbf{v}$. Form the complex solution:

$$e^{(a+bi)t}\mathbf{v}$$

Expanding complex solutions

- Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, expand:

$$e^{bit} = \cos bt + i\sin bt$$

- Multiply this into the eigenvector, and separate into real and imaginary parts $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.
- Form the general solution:

$$\boxed{\mathbf{x}_{\text{gen}} = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}}$$

Graphing solutions from complex eigenvalues

- To graph the solutions, ignore the e^{at} factor at first.
- Choose one solution and plug in values of t that make $bt = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.
- You should get ellipses.
- The factor of e^{at} makes the ellipses grow if $a > 0$ and shrink if $a < 0$.
- Use initial conditions, if given, to solve for c_1 and c_2.

Example I

Find the general solution to the following system:

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \mathbf{x}$$

$$\begin{aligned}
r &= 2 \pm i \\
r = 2 + i \quad &\Longrightarrow \quad \mathbf{v} = \begin{pmatrix} 1+i \\ -2 \end{pmatrix} \\
\mathbf{x}^{(1)} &= e^{(2+i)t} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} \\
&= e^{2t}(\cos t + i \sin t) \begin{pmatrix} 1+i \\ -2 \end{pmatrix} \\
&= e^{2t} \left[\begin{pmatrix} \cos t - \sin t \\ -2\cos t \end{pmatrix} + i \begin{pmatrix} \cos t + \sin t \\ -2 \sin t \end{pmatrix} \right] \\
\mathbf{x}_{\text{gen}} &= \boxed{c_1 e^{2t} \begin{pmatrix} \cos t - \sin t \\ -2\cos t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \cos t + \sin t \\ -2 \sin t \end{pmatrix}}
\end{aligned}$$

Example II

Graph some solution trajectories to the previous system of equations:

$$\mathbf{x}_{\text{gen}} = c_1 e^{2t} \begin{pmatrix} \cos t - \sin t \\ -2\cos t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \cos t + \sin t \\ -2\sin t \end{pmatrix}$$

Graph: Take $c_1 = 1, c_2 = 0$. Forget the e^{2t} for now. Graph $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. Get ellipse from $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. [Label values of t.]

$e^{2t} \Longrightarrow$ get outward swirls. [Graph on a separate graph.]

Example III

Find the general solution to the following system:

$$\mathbf{x}' = \begin{pmatrix} 3 & -5 \\ 5 & -5 \end{pmatrix} \mathbf{x}$$

$$r = -1 + 3i \Longrightarrow \mathbf{v} = \begin{pmatrix} 4 + 3i \\ 5 \end{pmatrix}$$

$$\boxed{\mathbf{x}_{\text{gen}} = c_1 e^{-t} \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 3\cos 3t + 4\sin 3t \\ 5\sin 3t \end{pmatrix}}$$

Example IV

Graph some solution trajectories to the previous system of equations:

$$\mathbf{x}_{\text{gen}} = c_1 e^{-t} \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 3\cos 3t + 4\sin 3t \\ 5\sin 3t \end{pmatrix}$$

$\begin{pmatrix} 4 \\ 5 \end{pmatrix} \to \begin{pmatrix} -3 \\ 0 \end{pmatrix} \to \begin{pmatrix} -4 \\ -5 \end{pmatrix} \to \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Ellipse oriented NE-SW, starting at $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and spiraling counterclockwise in to the origin.

Example V

Solve the following system and graph the solution trajectory:

$$\mathbf{x}' = \begin{pmatrix} 3 & -5 \\ 5 & -5 \end{pmatrix} \mathbf{x}, \mathbf{x}(0) = \begin{pmatrix} 6 \\ 15 \end{pmatrix}$$

$$\begin{aligned} \mathbf{x}_{\text{gen}} &= c_1 e^{-t} \begin{pmatrix} 4\cos 3t - 3\sin 3t \\ 5\cos 3t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 3\cos 3t + 4\sin 3t \\ 5\sin 3t \end{pmatrix} \\ & c_1 = 3, c_2 = -2 \\ & \boxed{\mathbf{x} = e^{-t} \begin{pmatrix} 6\cos 3t - 17\sin 3t \\ 15\cos 3t - 10\sin 3t \end{pmatrix}} \end{aligned}$$

Ellipse oriented NE-SW, starting at $\begin{pmatrix} 6 \\ 15 \end{pmatrix}$ and spiraling counterclockwise in to the origin.