XIII. Euler equations

Lesson Overview

- A differential equation of the form

$$x^2y'' + \alpha xy' + \beta y = 0$$

  is called an Euler equation.

- To solve it, solve the characteristic equation for r:

$$r^2 + (\alpha - 1)r + \beta = 0$$

- You might get two real roots, one repeated real root, or two complex conjugate roots.

- **Real, distinct roots**:

$$\boxed{y_{\text{gen}} = c_1x^{r_1} + c_2x^{r_2}}$$

- **Real, repeated roots**:

$$\boxed{y_{\text{gen}} = c_1x^{r_1} + c_2x^{r_1}\ln x}$$

- **Complex roots**: $r_1, r_2 = a \pm bi$

$$\boxed{y_{\text{gen}} = c_1x^a\cos\left(\ln x^b\right) + c_2x^a\sin\left(\ln x^b\right)}$$

Example I

Find the general solution to the differential equation:

$$x^2y'' - 3xy' + 3y = 0$$

$$\begin{aligned} r^2 - 4r + 3 &= 0 \\ (r-1)(r-3) &= 0 \Longrightarrow r = 1, 3 \\ y_{\text{gen}} &= \boxed{c_1 x + c_2 x^3} \end{aligned}$$

Example II

Find the general solution to the differential equation:

$$x^2 y'' - 7xy' + 16y = 0$$

$$\begin{aligned} r^2 - 8r + 16 &= 0 \\ (r-4)(r-4) &= 0 \Longrightarrow r = 4, 4 \\ y_{\text{gen}} &= \boxed{c_1 x^4 + c_2 x^4 \ln x} \end{aligned}$$

Example III

Find the general solution to the differential equation:

$$x^2 y'' - xy' + 5y = 0$$

$$\begin{aligned} r^2 - 2r + 5 &= 0 \Longrightarrow r = 1 \pm 2i \\ y_{\text{gen}} &= \boxed{c_1 x \cos \ln x^2 + c_2 x \sin \ln x^2} \end{aligned}$$

Example IV

Find the general solution to the differential equation:

$$x^2y'' - 6xy' + 12y = 0$$

$$\begin{aligned} r^2 - 7r + 12 &= 0 \\ (r-3)(r-4) &= 0 \implies r = 3, 4 \\ y_{\text{gen}} &= \boxed{c_1x^3 + c_2x^4} \end{aligned}$$

Example V

Find the general solution to the differential equation:

$$x^2y'' + 5xy' + 4y = 0$$

$$\begin{aligned} r^2 + 4r + 4 &= 0 \\ (r+2)(r+2) &= 0 \implies r = -2, -2 \\ y_{\text{gen}} &= \boxed{c_1x^{-2} + c_2x^{-2}\ln x} \end{aligned}$$

Example VI

Find the general solution to the differential equation:

$$x^2y'' - 3xy' + 29y = 0$$

$$\begin{aligned} r^2 - 4r + 29 &= 0 \implies r = 2 \pm 5i \\ y_{\text{gen}} &= \boxed{c_1x^2\cos\ln x^5 + c_2x^2\sin\ln x^5} \end{aligned}$$